The Lions domain decomposition algorithm on non matching cell-centered finite volume meshes

René Cautrès, Raphaèle Herbin, Florence Hubert

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Abstract

We propose a new finite volume scheme for convection diffusion equation on non matching grids. We give error estimates for H^2 solutions of the continuous problem. We then present a finite volume version of an adaptation of the Schwarz algorithm due to P.L. Lions, and prove, for a fixed mesh, its convergence towards the finite volume scheme on the whole domain. Numerical experiments illustrate the theoretical convergence order and the convergence of the Schwarz algorithm.

Keywords Domain decomposition, finite volume scheme, Schwarz algorithm, non matching grids.

AMS Subject Classification 65N12

1 Introduction

Let us consider the following diffusion-convection equation:

$$\begin{cases} -\Delta u + \operatorname{div}(\boldsymbol{v} u) + bu = f \quad \text{on} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2, 3$, $\boldsymbol{v} \in C^1(\Omega, \mathbb{R}^d)$, $b \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. The domain Ω is discretized with a grid which may feature some non matching cells, such as described in Figure 1.

We consider here the so-called "cell–centered" finite volume scheme, also sometimes called "finite volume–finite difference" scheme, where the discrete unknowns are located at some

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point in the control volumes and the normal fluxes to the cell boundaries are discretized by finite differences (see e. g. [Eymard *et al.* 2000]); we study the Lions method using Robin interface conditions (see [Lions 1990]) on this type of discretization.

The main interest of such a scheme is to use it when the domain Ω is decomposed into several nonoverlapping subdomains which may be meshed independently. This decomposition may then be used either as an iterative solver by itself or as a preconditioner in a conjugate gradient.

We recall that domain decomposition methods (see e.g. [Le Tallec 1994], or [Quarteroni & Valli 1999]) were introduced as a mean to perform large scale computations : thanks to the possibility of using a parallel computer, but also merely thanks to the efficiency of the iterative solver which may be associated to the domain decomposition. Most of the convergence studies were performed in the framework of finite element schemes, but recently finite volume schemes, which are widely used in industry, have also been studied [Achdou *et al.* 2002] [Cautrès *et al.* 2000a], [Cautrès *et al.* 2000b].

The case of non matching grids is of great importance: one would like to be able to mesh the subdomain independently from one another, and this usually results in non matching grids at the subdomain interface. In [Le Tallec & Sassi 1995], an augmented Lagragian method was introduced to deal with the problem. More recently the so-called "mortar method" was shown to give a precise approximation of second order elliptic problems in the framework of finite element methods [Bernardi *et al.* 1989], mixed finite element methods [Arbogast *et al.* 2000], finite volume element methods [Ewing *et al.* 2000], cell-centered finite volume methods [Achdou *et al.* 2002], [Saas *et al.* 2002].

Contrary to these methods, here we shall be much less cautious in the treatment at the subdomain interfaces in that we shall not try to adapt the fluxes on the non-matching interface. Hence we shall loose some precision on general meshes, but we gain in simplicity and in stability.

In the second section, we propose a finite volume scheme for Problem (1.1) which allows non consistent fluxes across atypical edges. This scheme was already successfully used in oil engineering applications [Aavatsmark *et al.* 2001], [Belmouhoub 1996]. Here we prove the convergence of the FV-scheme and a sharp error estimate under adequate assumptions on the unique weak solution to Problem (1.1). We only study here the case of non homogeneous Dirichlet boundary conditions, but Neumann and Robin conditions may also be considered with the technical tools developed in [Gallouët *et al.* 2000].

In the third section, we consider the decomposition of Ω in several non overlapping domains $(\Omega_i)_{i\in I}$ and use a discrete version of the Lions adaptation [Lions 1990] of the Schwarz algorithm in order to solve Problem (1.1): for a given $\alpha \in \mathbb{R}_+$, choose $u^{(0)} \in H_0^1(\Omega)$, and solve for each $n \geq 0$ and for each subdomain Ω_i, $i \in I$:

$$\begin{cases} -\Delta u_i^{(n+1)} + \operatorname{div}(\boldsymbol{v}\, u_i^{(n+1)}) + b u_i^{(n+1)} = f_i \;\text{ on }\; \Omega_i, \\ u_i^{(n+1)} = g_i \;\text{ on }\; \Gamma_i, \\ \dfrac{\partial u_i^{(n+1)}}{\partial n_i} + \alpha u_i^{(n+1)} = -\dfrac{\partial u_j^{(n)}}{\partial n_j} + \alpha u_j^{(n)} \;\text{ on }\; \gamma_{i,j}, \forall j \in I_i, \end{cases} \tag{1.2}$$

where $f_i = f_{|\Omega_i}$, $\Gamma_i = \partial\Omega_i \cap \partial\Omega$, $g_i = g_{|\Gamma_i}$, where $I_i = \{j \in I;\, j \neq i,\, \mathrm{m}(\overline{\Omega}_j \cap \overline{\Omega}_i) > 0\}$, where $\gamma_{i,j} \subset \partial\Omega_i$ is defined by $\gamma_{i,j} = \overline{\Omega_i} \cap \overline{\Omega_j}$ for all $j \in I_i$, and where n_i is the normal unit vector to $\gamma_{i,j}$ outward to Ω_i.

We present a finite volume version of this algorithm for which the convergence proof of P.L. Lions may be adapted. The convergence of such a scheme has first been proved in [Achdou *et al.* 2002] for a strong solution of Problem (1.1) for positive b. In [Cautrès *et al.* 2000b], we obtained the convergence for a weak solution of Problem (1.1) using techniques developped in [Eymard *et al.* 2000] for a general diffusion problem. We extend here the result in presence of a convection term.

We finally present in Section 5 numerical results in the case of decomposition of two, three, four and nine domains. We give numerical estimates of the error and emphasize, for a fixed mesh, the convergence of the iterative scheme towards the FV-scheme, the speed of convergence as a function of α.

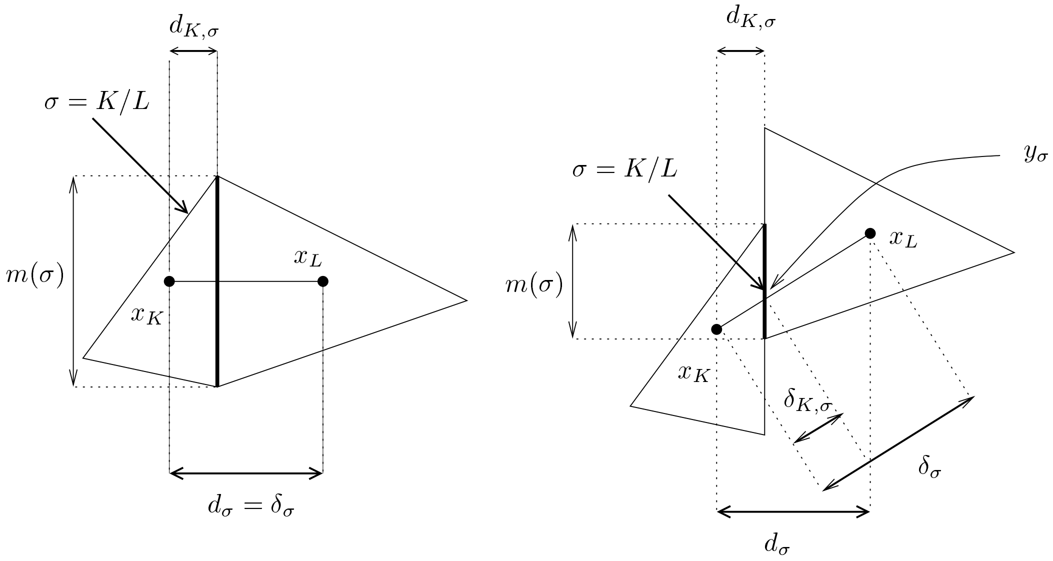


Figure 1: Notations for standard edges (left) and atypical edges (right)

2 The finite volume scheme

The cell–centered finite volume scheme was thoroughly presented and its convergence properties studied (in the case of conforming meshes) in several former papers. In the case of elliptic equations, L^2 error estimates were presented in [Herbin 1995] for triangular meshes, in [Mishev 1998] for Voronoï meshes, and finally in [Gallouët *et al.* 2000] for general meshes and boundary conditions. A convergence result without any assumption of regularity of the solution can be found in [Eymard *et al.* 1999], and an approximate gradient was constructed in [Eymard *et al.* 2001]. Noncoercive elliptic equations were also studied, for both regular H^{-1} and measure data [Droniou & Gallouët 2002], [Droniou *et al.* 2003]. Finally, a thorough study of finite volume schemes for linear or non-linear elliptic, parabolic and hyperbolic equations may be found in [Eymard *et al.* 2000], which we refer to for further details. We shall however need to recall here some basic principles, notations and definitions which were introduced in these papers, in order that the present one be self-contained.

The finite volume scheme is found by integrating equation (1.1) on a given control volume

K of a discretization mesh and finding an approximation of the normal fluxes, particularly on the interface σ of two control volumes K and L, namely $-\displaystyle\int_\sigma \nabla u(x)\cdot \mathbf{n}_{K,\sigma}\,dy(x)$ or $-\displaystyle\int_\sigma \nabla u(x)\cdot \mathbf{n}_{L,\sigma}\,dy(x)$, where $\mathbf{n}_{K,\sigma}$ (resp. $\mathbf{n}_{L,\sigma}$) is the normal unit vector to σ outward to K (resp. L) and dy is the integration symbol for the $(d-1)$-dimentional Lebesgue mesure on the hyperplane that contains the edge σ. The discretization of such a flux may be performed with a differential quotient involving values of the unknown located on the interface between two control volumes, on either side of this interface. The problem of the consistency of these fluxes at an interface σ, between two control volumes K and L which do not necessarily match, will be studied below.

Let us first give the following definition of an atypical mesh.

2.1 The meshes

Definition 2.1 (Finite volume meshes) *A finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are open polygonal (or polyhedral) convex subsets of Ω (with positive measure), a family of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, denoted by $\mathcal{E}$ (these are the edges (if $d=2$) or sides (if $d=3$) of the control volumes), with strictly positive $(d-1)$-dimensional measure, and a family of points of $\overline{\Omega}$ denoted by $\mathcal{P}$.*

The finite volume mesh is said to be "admissible" if properties (i) to (vi) below are satisfied, and "atypical" if properties (i) to (v) only are satisfied.

(i) The closure of the union of all the control volumes is $\overline{\Omega}$.

(ii) For any $K\in\mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \overline{K}\setminus K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$, and $\mathcal{E} = \cup_{K\in\mathcal{T}}\mathcal{E}_K$.

(iii) For any $(K,L)\in\mathcal{T}^2$ with $K\neq L$, either the $(d-1)$-dimensional Lebesgue measure of $\overline{K}\cap\overline{L}$ is 0 or $\overline{K}\cap\overline{L}=\overline{\sigma}$ for some $\sigma\in\mathcal{E}$, which will then be denoted by $K|L$.

(iv) The family $\mathcal{P}=(x_K)_{K\in\mathcal{T}}$ is such that for any $\sigma\in\mathcal{E}_{\rm int}$ such that $\sigma=K|L$, one has $x_Kx_L\cdot\mathbf{n}_{K,L}>0$, where $\mathbf{n}_{K,L}$ denotes the unit normal vector to $K|L$ outward to K.

(v) *For any* $\sigma \in \mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$, *let* K *be the control volume such that* $\sigma \in \mathcal{E}_K$. *If* $x_K \notin \sigma$, *let* $\mathcal{D}_{K,\sigma}$ *be the straight line going through* x_K *and orthogonal to* σ, *then the condition* $\mathcal{D}_{K,\sigma} \cap \sigma \neq \emptyset$ *is assumed; let* $y_\sigma = \mathcal{D}_{K,\sigma} \cap \sigma$.

(vi) *For any* $\sigma \in \mathcal{E}_{\text{int}}$ *such that* $\sigma = K|L$, *it is assumed that the straight line* $\mathcal{D}_{K,L}$ *going through* x_K *and* x_L *is orthogonal to* $K|L$.

In the sequel, the following notations are used. The mesh size is defined by: $h = \sup\{\mathrm{diam}(K),\ K \in \mathcal{T}\}$, *where* $\mathrm{diam}(K)$ *is the diameter of* $K \in \mathcal{T}$. *For any* $K \in \mathcal{T}$ *and* $\sigma \in \mathcal{E}$, $\mathrm{m}(K)$ *is the* d*-dimensional Lebesgue measure of* K *(i.e. area if* $d = 2$, *volume if* $d = 3$*),* $\mathrm{m}(\sigma)$ *the* $(d-1)$*-dimensional measure of* σ. *The set of interior (resp. boundary) edges is denoted by* $\mathcal{E}_{\text{int}}$ *(resp.* $\mathcal{E}_{\text{ext}}$*), that is* $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ *(resp.* $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$*). The set of neighbours of* K *is denoted by* $\mathcal{N}(K)$, *that is* $\mathcal{N}(K) = \{L \in \mathcal{T};\ \exists \sigma \in \mathcal{E}_K,\ \overline{\sigma} = \overline{K} \cap \overline{L}\}$. *Moreover, we shall distinguish the set* $\mathcal{E}_{\text{st}}$ *of "standard" edges belonging to* $\mathcal{E}_{\text{int}}$ *for which property (vi) is satisfied from the set* $\mathcal{E}_{\text{at}}$ *of "atypical" edges for which property (vi) is not satisfied. If* $\sigma = K|L \in \mathcal{E}_{\text{int}}$, *we denote by* δ_σ *or* $\delta_{K|L}$ *the Euclidean distance between* x_K *and* x_L *(which is positive), by* $d_{K,\sigma}$ *the distance from* x_K *to* σ, *by* $\delta_{K,\sigma}$ *the Euclidean distance from* x_K *to* y_σ, *where* y_σ *is the intersection between the straight line going through* x_K *and* x_L *and the hyperplane containing* $K|L$. *Define* $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$, *and note that if* $\sigma \in \mathcal{E}_{\text{st}}$ *and* $\sigma = K|L$, *then* $\delta_\sigma = d_\sigma$. *If* $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$, *let* d_σ *denote the Euclidean distance between* x_K *and* y_σ *(then* $d_\sigma = d_{K,\sigma}$*). For any* $\sigma \in \mathcal{E}$, *the "transmissibility" through* σ *is defined by* $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$ *if* $d_\sigma \neq 0$ *and* $\tau_\sigma = 0$ *if* $d_\sigma = 0$. *For simplicity, it is assumed that* $d_\sigma \neq 0$ *for any* $\sigma \in \mathcal{E}$.

Let us now present the regularity assumptions on the data.

Assumption 2.1 (Regularity of the data)

The domain Ω *is an open bounded polygonal subset of* $\mathbb{R}^d$, $d = 2, 3$. *Let* $f \in L^2(\Omega)$, $\boldsymbol{v} \in C^1(\Omega, \mathbb{R}^d)$, $b \in L^\infty(\Omega)$, $g \in H^{\frac{3}{2}}(\partial\Omega)$, *and* $\frac{1}{2}\mathrm{div}\, \boldsymbol{v}(x) + b(x) \geq 0$, *a.e.* x *in* Ω.

■

Note that the latter assumption could be weakened, see [Droniou & Gallouët 2002]. Under assumptions 2.1, Problem (1.1) has a unique variational solution in $H^1(\Omega)$.

2.2 Discretization of Problem (1.1)

Let $\mathcal{T}$ be an atypical mesh defined as in Definition 2.1. Each control volume K is associated to a discrete unknown u_K.

In order to obtain the discrete equations, integrate Equation (1.1) on each control volume K ;

$$\sum_{\sigma\in\mathcal{E}_K}\int_\sigma(-\nabla u(x)+u(x)\,\boldsymbol{v}(x)).\mathbf{n}_{K,\sigma}dy(x)+\int_K b(x)\,u(x)\,dx=\int_K f(x)\,dx.$$

For all $K\in\mathcal{T}$, and all $\sigma\in\mathcal{E}_K$, let us denote by $F_{K,\sigma}$ (resp. by $V_{K,\sigma}$) the approximate diffusion flux (resp. the approximate convection flux) that is to say, an approximation of $-\int_\sigma \nabla u(x)\cdot\mathbf{n}_{K,\sigma}\,dy(x)$ (resp. of $\int_\sigma u(x)\,\boldsymbol{v}(x).\mathbf{n}_{K,\sigma}\,dy(x)$).

A finite volume approximation of Problem (1.1) may then be written as the following linear system of equations:

$$\sum_{\sigma\in\mathcal{E}_K}F_{K,\sigma}+\sum_{\sigma\in\mathcal{E}_K}V_{K,\sigma}+b_K\,\mathrm{m}(K)\,u_K=\mathrm{m}(K)f_K,\ \forall K\in\mathcal{T},\tag{2.3}$$

where:

$$f_K=\frac{1}{\mathrm{m}(K)}\int_K f(x),\,dx\quad b_K=\frac{1}{\mathrm{m}(K)}\int_K b(x),\,dx\tag{2.4}$$

and where $F_{K,\sigma}$ (resp. $V_{K,\sigma}$) denotes the following approximation of the diffusion (resp. convection) flux $-\int_\sigma \nabla u(x)\cdot\mathbf{n}_{K,\sigma}\,dy(x)$ (resp. $\int_\sigma u(x)\,\boldsymbol{v}(x).\mathbf{n}_{K,\sigma}\,dy(x)$).

The <u>diffusive flux</u> is approximated by a central finite difference scheme:

$$F_{K,\sigma}=-\frac{\mathrm{m}(\sigma)}{\delta_{K,\sigma}}(u_\sigma-u_K),\tag{2.5}$$

the values u_σ being determined by the conservativity of the flux on the interior interfaces:

$$F_{K,\sigma}=-F_{L,\sigma}\text{ if }\sigma=K|L,\tag{2.6}$$

and by the boundary condition on the exterior edges:

$$u_\sigma=g(y_\sigma),\,\forall\sigma\in\mathcal{E}_{\text{ext}}.\tag{2.7}$$

The numerical convective flux $V_{K,\sigma}$ is obtained with a classical upstream scheme:

$$V_{K,\sigma} = \begin{cases} v_{K,\sigma}^{+}\, u_K - v_{K,\sigma}^{-}\, u_L \text{ if } \ \sigma = K|L, \\ \\ v_{K,\sigma}^{+}\, u_K - v_{K,\sigma}^{-}\, g_\sigma \text{ if } \ \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}, \end{cases} \tag{2.8}$$

where

$$v_{K,\sigma} = \int_\sigma \boldsymbol{v}(x) \cdot \mathbf{n}_{K,\sigma}\, dy(x), \tag{2.9}$$

$v^+ = \max(v,0)$ and $v^- = -\min(v,0)$.

Remark 2.1 *Note that the regularity assumption $g \in H^{\frac{3}{2}}(\partial\Omega)$ is needed in (2.7) ; if g is only assumed to be a function of $L^2(\partial\Omega)$, then (2.7) may be replaced by $u_\sigma = \dfrac{1}{\mathrm{m}(\sigma)} \displaystyle\int_\sigma g(x)\, dy(x)$. However, the regularity $g \in H^{\frac{3}{2}}(\partial\Omega)$ must be assumed in order to obtain an error estimate, otherwise one only gets a convergence result as in [Eymard et al. 1999].*

Remark 2.2 *Note that the unknowns $(u_\sigma)_{\sigma\in\mathcal{E}_{\mathrm{int}}}$ may be eliminated by using (2.5) and (2.6), namely*

$$F_{K,\sigma} = \mathrm{m}(\sigma)\, \frac{u_L - u_K}{\delta_\sigma},$$

for $\sigma \in \mathcal{E}_{\mathrm{int}}$ and $\sigma = K|L$.

2.3 Discrete norms

Let us now introduce the space of piecewise constant functions associated with an atypical mesh and some "discrete H_0^1" norm for this space. This discrete norm will be used in the sequel to obtain some estimates on the approximate solution given by the finite volume scheme and to prove the convergence of the discrete solution to the exact solution to Problem (1.1), as h tends to 0.

Definition 2.2 (Discrete H_0^1 norm) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and $\mathcal{T}$ an admissible mesh. Let $X(\mathcal{T})$ be the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh. For $u_\mathcal{T} \in X(\mathcal{T})$ such that $u_\mathcal{T}(x) = u_K$ a.e. in K, for all $K \in \mathcal{T}$, define the discrete H_0^1 norm of $u_\mathcal{T}$ by:*

$$\|u_\mathcal{T}\|_{1,\mathcal{T}} = \left(\sum_{\sigma \in \mathcal{E}} \mathrm{m}(\sigma)\, \delta_\sigma \, \left(\frac{D_\sigma u_\mathcal{T}}{\delta_\sigma} \right)^2 \right)^{\frac{1}{2}}, \tag{2.10}$$

where, for any $\sigma \in \mathcal{E}$,

$$D_\sigma u_\mathcal{T} = \begin{cases} |u_K - u_L| & \text{if } \sigma \in \mathcal{E}_{\mathrm{int}} \text{ and } \sigma = K|L, \\ |u_K| & \text{if } \sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K, \end{cases}$$

where u_K denotes the value taken by $u_\mathcal{T}$ on the control volume K and the sets $\mathcal{E}$, $\mathcal{E}_{\mathrm{int}}$, $\mathcal{E}_{\mathrm{ext}}$ and $\mathcal{E}_K$ are defined in Definition 2.1.

We now extend a discrete Poincaré inequality, which was proven in [Eymard *et al.* 2000] in the case of admissible meshes, to the case of "atypical" meshes, under the more restrictive assumption of quasi–uniformness of the mesh.

Lemma 2.1 (Discrete Poincaré inequality) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, $\mathcal{T}$ an atypical finite volume mesh of Ω in the sense of Definition 2.1.*
Let $\eta > 0$ such that $\eta\, h^d \leq \mathrm{m}(K)$, $\forall K \in \mathcal{T}$. Let $u_\mathcal{T} \in X(\mathcal{T})$, then

$$\|u_\mathcal{T}\|_{L^2(\Omega)} \leq C_\Omega \|u_\mathcal{T}\|_{1,\mathcal{T}}, \tag{2.11}$$

where C_Ω only depends on Ω, d and η.

Proof

Let $u_\mathcal{T} \in X(\mathcal{T})$ with $u_\mathcal{T}(x) = u_K$, a.e. $x \in K$, for all $K \in \mathcal{T}$. Following [Gallouët *et al.* 2000] or [Eymard *et al.* 2000], for $\sigma \in \mathcal{E}$, define χ_σ from $\mathbb{R}^d \times \mathbb{R}^d$ to $\{0, 1\}$ by $\chi_\sigma(x, y) = 1$ if $\sigma \cap [x, y] \neq \emptyset$ and $\chi_\sigma(x, y) = 0$ otherwise.

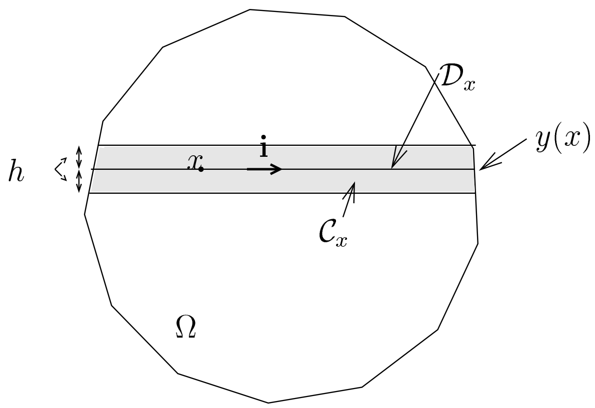


Figure 2: Notations

Let $u \in X(\mathcal{T})$. Let $\mathbf{i}$ be a given unit vector. For all $x \in \Omega$, let $\mathcal{D}_x$ be the semi-line defined by its origin, x, and the vector $\mathbf{i}$. Let $y(x)$ such that $y(x) \in \mathcal{D}_x \cap \partial\Omega$ and $[x, y(x)] \subset \overline{\Omega}$, where $[x, y(x)] = \{tx + (1-t)y(x),\, t \in [0,1]\}$ (i.e. $y(x)$ is the first point where $\mathcal{D}_x$ meets $\partial\Omega$). Let $K \in \mathcal{T}$; thanks to the fact that $\mathbf{i}$ is fixed and that the number of edges is finite, we may write that for a.e. $x \in K$, one has

$$|u_K| \leq \sum_{\sigma\in\mathcal{E}} D_\sigma u \, \chi_\sigma(x, y(x)).$$

By the Cauchy Schwarz inequality, the above inequality yields

$$|u_{\mathcal{T}}(x)|^2 \leq \left(\sum_{\sigma\in\mathcal{E}} \frac{|D_\sigma u_{\mathcal{T}}|^2}{\delta_\sigma} \chi_\sigma(x, y(x)) \right) \left(\sum_{\sigma\in\mathcal{E}} \delta_\sigma \, \chi_\sigma(x, y(x)) \right). \tag{2.12}$$

Let us prove that:

$$\sum_{\sigma\in\mathcal{E}} \delta_\sigma \, \chi_\sigma(x, y(x)) \leq \frac{\operatorname{diam}(\Omega)\, C_d}{\eta}, \tag{2.13}$$

where $C_d \geq 0$ only depending on d. Since $\delta_\sigma \leq 2\,h$ for all $\sigma \in \mathcal{E}$, it is sufficient to prove that

$$\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x, y(x)) \leq \frac{\operatorname{diam}(\Omega)\, C_d}{2\,\eta\, h}. \tag{2.14}$$

Remark that $\sum_{\sigma\in\mathcal{E}} \chi_\sigma(x, y(x))$ is the number of edges intersected by $\mathcal{D}_x$. Let $x \in \Omega$ be such that $\mathcal{D}_x$ does not contains any edge. Define $\mathcal{C}_x$ by $\mathcal{C}_x = \{z \in \Omega,\, d(z, \mathcal{D}_x) < h\}$, as shown in Figure 2. If a control volume K has an edge which intersects $\mathcal{D}_x$, since $\mathrm{diam}(K) \le h$, then $K \subset \mathcal{C}_x$. Now, if $\mathcal{D}_x$ intersects a control volume, it intersects exactly two edges of this control volume, for a.e $x \in \Omega$, because of the convexity of K. Hence

$$\sum_{\sigma \in \mathcal{E}} \chi_\sigma(x, y(x)) \le \mathrm{card}\{K \in \mathcal{T};\ K \subset \mathcal{C}_x\}.$$

Now since

$$\sum_{K \subset \mathcal{C}_x} \mathrm{m}(K) \le \mathrm{m}(\mathcal{C}_x) \le \mathrm{diam}(\Omega)\, C_1\, h^{d-1},$$

with $C_1 \ge 0$ only depending on d, using the assumption $\eta\, h^d \le \mathrm{m}(K)$ for all $K \in \mathcal{T}$, we obtain:

$$\mathrm{card}\{K \in \mathcal{T};\ K \subset \mathcal{C}_x\} = \sum_{K \subset \mathcal{C}_x} 1 \le \frac{\mathrm{diam}(\Omega)\, C_1}{\eta\, h}. \tag{2.15}$$

Using now (2.15), we deduce (2.14) and then (2.13) with $C_d = 2\, C_1$.

From (2.12) and (2.13) , we obtain

$$|u_\mathcal{T}(x)|^2 \le \frac{\mathrm{diam}(\Omega)\, C_d}{\eta} \left(\sum_{\sigma \in \mathcal{E}} \frac{|D_\sigma u_\mathcal{T}|^2}{\delta_\sigma} \chi_\sigma(x, y(x)) \right),$$

for *a.e.* $x \in K$, for all $K \in \mathcal{T}$.

Integrating over Ω, and noting that $\displaystyle\int_\Omega \chi_\sigma(x, y(x))\, dx \le \mathrm{diam}(\Omega)\, \mathrm{m}(\sigma)$, yields (2.11) with C_Ω only depending on Ω, d and η. ■

The discrete H_0^1-norm may then be used to prove existence and uniqueness of the solution to Problem (2.3)-(2.9) (see [Eymard *et al.* 2000] or [Gallouët *et al.* 2000]):

Theorem 2.1 (Existence and uniqueness)

Under Assumptions 2.1, Problem (2.3)-(2.9) has a unique solution.

Another important property which was proven in [Gallouët *et al.* 2000] and which also holds here, is the maximum principle stated below. This property may be of utmost importance in applications where the physical bounds of some quantity (say, a concentration) must be respected by the numerical approximation.

Theorem 2.2 (Maximum principle) *Under Assumption 2.1, let $\mathcal{T}$ be an admissible mesh in the sense of Definition 2.1, and assume that the values $(f_K)_{K\in\mathcal{T}}$ and $(u_\sigma)_{\sigma\in\mathcal{E}_{\rm ext}}$ computed in (2.6) and (2.7) are nonnegative. Then the solution $(u_K)_{K\in\mathcal{T}}$ to (2.3)-(2.9) satisfies $u_K \geq 0$ for all $K \in \mathcal{T}$.*

The proof of this maximum principle is given in [Eymard *et al.* 2000] or [Gallouët *et al.* 2000] for various types of boundary conditions on a finite volume admissible mesh.
It is straightforward to adapt it to the non-standard case. In fact, one of the main arguments of the proof is that the diffusion term is discretized under the form

$$\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} \tau_\sigma(u_K - u_L) + \sum_{\sigma\in\mathcal{E}_{\rm ext},\sigma\in\mathcal{E}_K} \tau_\sigma(u_K - u_\sigma), \tag{2.16}$$

with positive "transmissibilities" τ_σ. Hence the maximum principle would hold for any positive value for τ_σ (but then of course, the choice of τ_σ is also important for the consistency of the flux, and therefore for the convergence of the scheme).

3 Error estimate

In this section, we prove a sharp error estimate between the approximate solution $u_{\mathcal{T}}$ to (2.3)-(2.4) and the exact solution u to (1.1), assuming $u \in C^2(\overline{\Omega})$ or $u \in H^2(\Omega)$ and quasi-uniformness of the mesh, if atypical. This is a generalization of some of the results of [Herbin 1995], [Eymard *et al.* 2000] or [Gallouët *et al.* 2000] to atypical meshes. The main novelty with respect to these previous works consists in controlling the consistency error on the fluxes on atypical edges.

Lemma 3.1 (Consistency errors) *Let $u \in H^2(\Omega)$ be the unique variational solution to Problem (1.1), and let $\mathcal{T}$ be an atypical mesh in the sense of Definition 2.1. For a given control volume K and for $\sigma \in \mathcal{E}_K$, let the exact diffusion (resp. convection) flux $\overline{F}_{K,\sigma}$ (resp. $\overline{V}_{K,\sigma}$) through σ outward to K be defined by:*

$$\overline{F}_{K,\sigma} = -\int_\sigma \nabla u(x)\cdot \mathbf{n}_{K,\sigma} dy(x) \qquad \text{and} \qquad \overline{V}_{K,\sigma} = \int_\sigma u(x)\, \boldsymbol{v}(x)\cdot \mathbf{n}_{K,\sigma} dy(x). \tag{3.17}$$

Let $F^\star_{K,\sigma}$ and $V^\star_{K,\sigma}$ be defined by:

$$F^\star_{K,\sigma} = -m(\sigma)\,\frac{u(x_L)-u(x_K)}{\delta_\sigma},\ \textit{if}\ \sigma = K|L, \tag{3.18}$$

and

$$F^\star_{K,\sigma} = -m(\sigma)\,\frac{u(y_\sigma)-u(x_K)}{\delta_{K,\sigma}},\ \textit{if}\ \sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K, \tag{3.19}$$

and

$$V^\star_{K,\sigma} = \begin{cases} v^+_{K,\sigma}\,u(x_K) - v^-_{K,\sigma}\,u(x_L),\ \textit{if}\ \sigma = K|L, \\ \\ v^+_{K,\sigma}\,u(x_K) - v^-_{K,\sigma}\,u(y_\sigma),\ \textit{if}\ \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}. \end{cases} \tag{3.20}$$

Then, the consistency error on the diffusion and convection flux are defined as

$$R_{K,\sigma} = \frac{1}{\mathrm{m}(\sigma)}\left(\overline{F}_{K,\sigma} - F^\star_{K,\sigma}\right)\ \textit{and}\ r_{K,\sigma} = \frac{1}{\mathrm{m}(\sigma)}\left(\overline{V}_{K,\sigma} - V^\star_{K,\sigma}\right). \tag{3.21}$$

Moreover, we define

$$\rho_K = \frac{1}{m(K)}\int_K b(x)\,(u(x)-u(x_K))\,dx,\ \textit{for all}\ K \in \mathcal{T}. \tag{3.22}$$

Under Assumptions 2.1, let $\zeta > 0$, such that $\zeta\, diam(K) \le d_{K,\sigma}$, $\forall \sigma \in \mathcal{E} \cap \mathcal{E}_K$, $\forall K \in \mathcal{T}$. There exists C_1, only depending on d and ζ, and C_2, only depending on d, $\boldsymbol{v}$, ζ and d such that for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}_K$,

$$|R_{K,\sigma}| \le \begin{cases} C_1\,h\,(\mathrm{m}(\sigma)\,\delta_\sigma)^{-\frac{1}{2}}\|u\|_{H^2(\mathcal{V}_\sigma)},\ \textit{if}\ \sigma \in \mathcal{E}_{\mathrm{st}} \cup \mathcal{E}_{\mathrm{ext}}, \\ \\ C_1\,h\,(\mathrm{m}(\sigma)\,\delta_\sigma)^{-\frac{1}{2}}\|u\|_{H^2(\mathcal{V}_\sigma)} + 2\,\mathrm{m}(\sigma)^{-\frac{1}{2}}\,\|\nabla u\|_{(L^2(\sigma))^d},\ \ \textit{if}\ \sigma \in \mathcal{E}_{\mathrm{at}}, \end{cases} \tag{3.23}$$

$$|r_{K,\sigma}| \le C_2\,h\ (\mathrm{m}(\sigma)\,d_\sigma)^{-\frac{1}{p}}\ \|u\|_{W^{1,p}(\mathcal{V}_\sigma)}, \tag{3.24}$$

$$|\rho_K| \le \|b\|_{L^\infty(\Omega)}\,h\,m(K)^{-\frac{1}{p}}\ \|u\|_{W^{1,p}(K)}, \tag{3.25}$$

for all $p > d$ such that $p < +\infty$ if $d = 2$ and $p \le 6$ if $d = 3$, with $\mathcal{V}_{K,\sigma} = \{tx_K + (1-t)x$, for all $x \in \sigma$ and $t \in [0,1]\}$, $\mathcal{V}_\sigma = \mathcal{V}_{K,\sigma} \cup \mathcal{V}_{L,\sigma}$ if $\sigma \in \mathcal{E}_{\mathrm{int}}, \sigma = K|L$ (see Figure 3), and $\mathcal{V}_\sigma = \mathcal{V}_{K,\sigma}$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$.

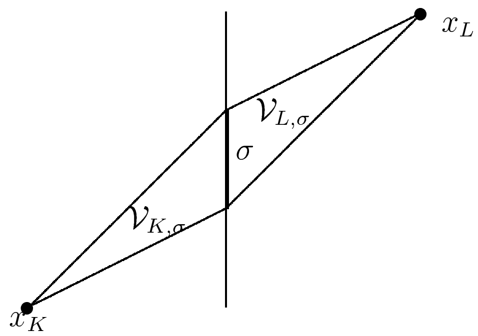


Figure 3: The domain $\mathcal{V}_\sigma$

Remark 3.1 (Sobolev's imbeddings)
Note that if $u \in H^2(\Omega)$, $\nabla u \in \left(H^1(\Omega)\right)^d$, so the trace of $|\nabla u|$ on $\sigma \in \mathcal{E}$ is well defined and is in $L^2(\sigma)$. Thanks to Sobolev's imbeddings, $u \in W^{1,p}(\Omega)$ for all p such that $1 \leq p < +\infty$ if $d = 2$ and such that $1 \leq p \leq 6$ if $d = 3$. Then (3.23), (3.24) and (3.25), are well defined.

Proof of Lemma 3.1
The consistency error on the diffusion flux on standard edges (3.23) ($\sigma \in \mathcal{E}_{\text{ext}} \cup \mathcal{E}_{\text{st}}$) is thoroughly dealt with in [Gallouët *et al.* 2000] or [Eymard *et al.* 2000]. The proof of (3.24) and (3.25) was performed in [Gallouët *et al.* 2000] for standard edges, but readily extends to atypical edges (3.24). Hence there only remains to prove (3.23). Let $\sigma \in \mathcal{E}_{\text{at}}$, with $\sigma = K|L$. Since the restriction of u to $\mathcal{V}_\sigma$ belongs to $H^2(\mathcal{V}_\sigma)$, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of elements of $C^2(\overline{\mathcal{V}_\sigma})$ which converges to u in $H^2(\mathcal{V}_\sigma)$. Thanks to Sobolev's imbeddings, we may prove (3.23) with φ_n instead of u.
Following [Gallouët *et al.* 2000], a Taylor expansion using $\varphi_n \in C^2(\overline{\mathcal{V}_\sigma})$ and an integration on σ yield: $|R_{K,\sigma}| \leq B_{K,\sigma} + B_{L,\sigma} + B_{K,L}$ with

$$\begin{aligned} B_{M,\sigma} &= \frac{C_3}{\mathrm{m}(\sigma)\,\delta_\sigma} \int_\sigma \int_0^1 \|H(\varphi_n)\,(tx + (1-t)\,x_M)\|\,|x_M - x|^2\,t\,dt\,dy(x), \quad \text{for } M = K \text{ or } L \\ \text{and } B_{K,L} &= \frac{1}{\mathrm{m}(\sigma)} \left| \int_\sigma \nabla\varphi_n(x) \cdot (\tilde{\mathbf{n}}_{K,L} - \mathbf{n}_{K,\sigma})\,dy(x) \right|, \end{aligned} \tag{3.26}$$

where $C_3 \geq 0$ only depending on d, where $H(\varphi_n)(z)$ denotes the Hessian matrix of φ_n at point z and $\|H(\varphi_n)(z)\|^2 = \sum_{i,j=1}^{d} |D_iD_j\varphi_n(z)|^2$.

Let us now deal with the term $B_{K,\sigma}$. Using the same technique as that of [Gallouët *et al.* 2000] (proof of Lemma 3.3), noting that $\mathrm{m}(\mathcal{V}_{K,\sigma}) = \dfrac{\mathrm{m}(\sigma) d_{K,\sigma}}{d}$, for $d=2$ or $d=3$, we find that:

$$B_{K,\sigma} \leq \frac{C_4 \operatorname{diam}(K)^2}{(\mathrm{m}(\sigma)\,\delta_\sigma)^{\frac{1}{2}}\,(d_{K,\sigma}\delta_\sigma)^{\frac{1}{2}}} \left(\int_{\mathcal{V}_{K,\sigma}} \|H(\varphi_n)(z)\|^2 dz \right)^{\frac{1}{2}},$$

where $C_4 \geq 0$ only depending on d. Hence, using the fact that $\delta_\sigma \geq d_{K,\sigma} \geq \zeta \operatorname{diam}(K)$ and $\operatorname{diam}(K) \leq h$, we obtain:

$$B_{K,\sigma} \leq \frac{C_5\, h}{\zeta\,(\mathrm{m}(\sigma)\,\delta_\sigma)^{\frac{1}{2}}} \|H(\varphi_n)\|_{L^2(\mathcal{V}_{K,\sigma})}, \tag{3.27}$$

for some $C_5 \geq 0$, only depending on d. The same estimate holds for $B_{L,\sigma}$. Let us now deal with the term $B_{K,L}$ defined by (3.26). Since $\tilde{\mathbf{n}}_{K,L}$ and $\mathbf{n}_{K,\sigma}$ are unit vectors, applying the Cauchy-Schwarz inequality yields

$$B_{K,L} \leq 2\,\mathrm{m}(\sigma)^{-\frac{1}{2}} \|\nabla\varphi_n\|_{(L^2(\sigma))^d}. \tag{3.28}$$

The estimates (3.27) and (3.28) on $B_{K,\sigma}$, $B_{L,\sigma}$ and $B_{K,L}$ yield (3.23), for φ_n instead of u, for some $C_1 \geq 0$, only depending on d and ζ. We conclude by density the proof of Lemma 3.1. ∎

Theorem 3.1 (Error estimate)
Under Assumptions 2.1, let $\mathcal{T}$ be an atypical mesh in the sense of Definition 2.1, and let $\zeta > 0$ such that

$$\zeta\, h \leq d_{K,\sigma},\ \forall \sigma \in \mathcal{E} \cap \mathcal{E}_K,\ \forall K \in \mathcal{T}. \tag{3.29}$$

Let $u_{\mathcal{T}}$ be the unique solution to problem (2.3)-(2.9). Assume that the unique variational solution u to (1.1) belongs to $H^2(\Omega)$. Let $e_{\mathcal{T}}$ be defined by $e_{\mathcal{T}}(x) = e_K = u(x_K) - u_K$ a. e. $x \in K$, $K \in \mathcal{T}$. Then, there exists $\underline{C}$, (resp. $\overline{C} > 0$), only depending on Ω and μ (resp. u, $\boldsymbol{v}$, b, d, Ω and ζ) such that ,

$$\underline{C}\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq \|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq \overline{C} \left(h + \left(\sum_{\sigma \in \mathcal{E}_{\mathrm{at}}} \|\nabla u_{|\sigma}\|^2_{(L^2(\sigma))^d} \right)^{\frac{1}{2}} h^{\frac{1}{2}} \right), \tag{3.30}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete H_0^1 norm defined in Definition 2.2 and $\mathcal{E}_{\mathrm{at}}$ is the set of atypical interfaces defined in Definition 2.1.

Remark 3.2 *If the family of points $(x_K)_{K\in\mathcal{T}}$ satisfies $x_K \in K$, then, under Assumption (3.29), each ball with radius $\zeta\, h$ is included in K and there exists $\eta > 0$ only depending on d and ζ, such that $\eta\, h^d \le \mathrm{m}(K)$, $\forall K \in \mathcal{T}$. Furthermore we have $\zeta\,\mathrm{diam}(K) \le d_{K,\sigma}$. Hence Assumption (3.29) allows the use of the Poincaré inequality (2.11) with a constant C_Ω depending on Ω, d and ζ and the use of consistency errors of Lemma 3.1, even with the choice δ_σ rather than d_σ in the scheme (see Remark 3.3).*

The following corollaries are direct consequences of Theorem 3.1 and are of interest in the case of domain decomposition.

Corollary 3.1 ($W^{1,\infty}$ regularity)
Under the same assumptions as in Theorem 3.1, assume that $\nabla u \in (L^\infty(\Omega))^d$ and that for some $C_{\mathrm{at}} \ge 0$, not depending on $\mathcal{T}$,

$$\mathrm{card}(\mathcal{E}_{\mathrm{at}}) \le \frac{C_{\mathrm{at}}}{h^{d-1}} \text{ for } d = 2, \quad \text{or } d = 3. \tag{3.31}$$

Then , there exists $\underline{C}$, (resp. $\overline{C} > 0$), only depending on Ω and ζ (resp. u, $\boldsymbol{v}$, b, d, Ω and ζ) such that ,

$$\underline{C}\|e_{\mathcal{T}}\|_{L^2(\Omega)} \le \|e_{\mathcal{T}}\|_{1,\mathcal{T}} \le \overline{C}\ \left(h + (C_{\mathrm{at}})^{\frac{1}{2}}\ h^{\frac{1}{2}}\right). \tag{3.32}$$

Corollary 3.2 (Non overlapping domain decomposition)
Under the same assumptions as in Theorem 3.1, assume that Ω is decomposed into several non overlapping domains $(\Omega_i)_{i\in I}$ and there exists $\gamma \subset \bigcup_{i\in I} \partial\Omega_i$, with positive $\mathbb{R}^{d-1}$ measure, such that $\sigma \subset \gamma$ for all $\sigma \in \mathcal{E}_{\mathrm{at}}$, then there exists $\underline{C}$, (resp. $\overline{C} > 0$), only depending on Ω and ζ (resp. u, $\boldsymbol{v}$, b, d, Ω and ζ) such that

$$\underline{C}\|e_{\mathcal{T}}\|_{L^2(\Omega)} \le \|e_{\mathcal{T}}\|_{1,\mathcal{T}} \le \overline{C}\ \left(h + \|\nabla u_{|\gamma}\|_{(L^2(\gamma))^d}\, h^{\frac{1}{2}}\right). \tag{3.33}$$

Proof of Theorem 3.1 The left inequality in (3.30) is a direct consequence of the Poincaré inequality (see Lemma (2.1) and Remark (3.2)).

The proof of the right inequality in (3.30) closely follows that of Theorem 3.2 in [Gallouët *et al.* 2000], and we shall therefore only dwell on the difference introduced by the non-consistency of the diffusion flux through the non-standard edges. Integrating the first equation of system (1.1) over each control volume K, substracting (2.3) off the result, multiplying by e_K and summing over $K \in \mathcal{T}$ yields:

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 + T_C + T_b \le T_\rho + T_r + T_R, \tag{3.34}$$

where

$$T_C = \begin{cases} \displaystyle\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} (v_{K,\sigma}^+ \, e_K - v_{K,\sigma}^- \, e_L) \, e_K, \text{ if } \sigma = K|L, \\ \\ \displaystyle\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}^+ \, e_K \text{ if } \sigma \in \mathcal{E}_{\text{ext}}, \end{cases}$$

$T_b = \displaystyle\int_\Omega b(x) \; (e_{\mathcal{T}}(x))^2 \; dx$, $T_\rho = -\displaystyle\sum_{K\in\mathcal{T}} \mathrm{m}(K) \, \rho_K \, e_K$, $T_r = -\displaystyle\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \, r_{K,\sigma} \, e_K$ and $T_R = -\displaystyle\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \, R_{K,\sigma} \, e_K$. The control of the terms T_C, T_b T_ρ and T_r may be performed in the exact same way as in [Gallouët *et al.* 2000], and leads to:

$$\begin{cases} T_C + T_b \ge 0, \\ \\ T_\rho \le \dfrac{\varepsilon}{2} \|e_{\mathcal{T}}\|_{L^2(\Omega)}^2 + \dfrac{C_6}{\varepsilon} \, h^2, \text{ for all } \varepsilon > 0, \\ \\ T_r \le \dfrac{1}{4} \|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 + C_7 \, h^2 \end{cases} \tag{3.35}$$

where $C_6 \ge 0$ only depends on b, u and Ω, and $C_7 \ge 0$, only depends on u, $\boldsymbol{v}$, d, ζ and Ω. Let us now turn to the diffusion term T_R. Again, following [Gallouët *et al.* 2000], we introduce $R_\sigma = |R_{K,\sigma}|$, using the conservativity property of the scheme (2.6), reordering the summation over the edges, using Young's inequality, and the consistency error (3.23), one obtains:

$$T_R \leq \frac{1}{4}\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 + C_1^2\, h^2\, \|u\|_{H^2(\Omega)}^2 + 4 \sum_{\sigma\in\mathcal{E}_{\rm at}} \delta_\sigma\, \|\nabla u_{|\sigma}\|_{(L^2(\sigma))^2}^2.$$

Noting that $\delta_\sigma \leq 2\,h$, we deduce that there exists $C_8 \geq 0$, only depending on u, Ω, ζ such that

$$T_R \leq \frac{1}{4}\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 + C_8\, \left(h^2 + \left(\sum_{\sigma\in\mathcal{E}_{\rm at}} \|\nabla u_{|\sigma}\|_{(L^2(\sigma))^2}^2 \right) h \right). \tag{3.36}$$

From (3.34), (3.35) and (3.36), one has, for all $\varepsilon > 0$:

$$\frac{1}{2}\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \frac{\varepsilon}{2}\|e_{\mathcal{T}}\|_{L^2(\Omega)}^2 + C_9 \left(h^2 + \left(\sum_{\sigma\in\mathcal{E}_{\rm at}} \|\nabla u_{|\sigma}\|_{(L^2(\sigma))^d}^2 \right) h. \right) \tag{3.37}$$

where $C_9 \geq 0$ only depends on u, $\boldsymbol{v}$, b, d, Ω, ζ, and ε. Taking $\varepsilon = \dfrac{1}{2\,C_\Omega^2}$, where C_Ω is the Poincaré constant in (2.11), yields (3.30). Note that C_Ω depends only on d, Ω and ζ.

Remark 3.3 (d_σ instead of δ_σ)
In the approximation of the diffusive flux (2.5), we used $\delta_\sigma = |x_K - x_L|$ if $\sigma = K|L$, which is a natural choice. But in fact, one could also think of using d_σ instead of δ_σ (recall that $d_\sigma = \delta_\sigma$ for a standard edge). Then one should also use d_σ instead of δ_σ in the definition of the discrete H_0^1-norm (2.10). It can be proven that the above error estimate still holds in this case. Moreover, we implemented this new scheme and got numerical results which were extremely close to those obtained for the original scheme, which are presented in Section 5.
Let us briefly outline the modifications which are induced by this new choice in the proofs of the previous propositions.
In the proof of the Poincaré inequality, following [Eymard et al. 2000], one writes (2.12) with d_σ instead of δ_σ. Remarking that $d_\sigma \leq 2\,h$ and proceeding like in the previous proof, yields (2.11) with $C_\Omega > 0$ only depending on Ω, d and η.
Let us now turn to the consistency error: nothing changes for the standard and exterior

edges, while if $\sigma \in \mathcal{E}_{\rm at}$*, using* φ_n *instead of* u*, then* $|R_{K,\sigma}| \le B_{K,\sigma} + B_{L,\sigma} + B_{K,L}$*, with*

$$B_{M,\sigma} = \frac{C_3}{\mathrm{m}(\sigma)\, d_\sigma} \int_\sigma \int_0^1 \|H(\varphi_n)(tx + (1-t)x_M\| \, |x_M - x|^2 \, t \, dt \, dy(x), \quad \textit{for } M = K \textit{ or } L$$

and

$$B_{K,L} = \frac{1}{\mathrm{m}(\sigma)\, d_\sigma} \left| \int_\sigma \nabla\varphi_n(x) \cdot (\delta_\sigma \, \tilde{\boldsymbol{n}}_{K,L} - d_\sigma \, \mathbf{n}_{K,\sigma}) \, dy(x) \right|.$$

Hence $B_{M,\sigma}$ *now has the same expression as in [Gallouët et al. 2000], and we have:*

$$B_{K,\sigma} \le \frac{C_5 \, h}{\zeta \, (\mathrm{m}(\sigma)\, d_\sigma)^{\frac{1}{2}}} \, \|H(\varphi_n)\|_{L^2(\mathcal{V}_{K,\sigma})}.$$

Hence, using Cauchy-Schwarz inequality for $B_{K,L}$ *and noting that* $d_\sigma \ge \zeta(\mathrm{diam}(K) + \mathrm{diam}(L))$ *and* $d_\sigma \le \delta_\sigma \le \mathrm{diam}(K) + \mathrm{diam}(L)$*, we get that*

$$|R_{K,\sigma}| \le C_1 \, h \, (\mathrm{m}(\sigma)\, d_\sigma)^{-\frac{1}{2}} \|u\|_{H^2(\mathcal{V}_\sigma)} + \frac{2}{\zeta} \, \mathrm{m}(\sigma)^{-\frac{1}{2}} \, \|\nabla u\|_{(L^2(\sigma))^d},$$

where $C_1 \ge O$ *depends on* u*,* Ω*,* d*, and* ζ*.*
Hence we obtain (3.35), (3.36) and (3.37) with C_6*,* C_7*,* C_8 *and* C_9 *depending on* d*,* Ω*,* $\boldsymbol{v}$*,* g*,* u*,* b *and* ζ*. Hence the error estimates (3.32) also holds if* d_σ *is used rather than* δ_σ*.*

■

4 The finite volume Schwarz Algorithm

In this section we present a discrete domain decomposition algorithm which is the finite volume version of the Schwarz algorithm (1.2). We then prove, for a given atypical mesh $\mathcal{T}$, the convergence of this algorithm, under the following assumptions, towards the unique solution of Problem (2.3)-(2.9), i.e. the finite volume approximation to Problem (1.1).

Assumption 4.1

The domain Ω *is a path-connected bounded open polygonal (or polyhedral) subset of* $\mathbb{R}^d$*,* $d = 2, 3$*, which is decomposed into several non overlapping subdomains* $(\Omega_i)_{i\in I}$*, such that the interface* $\overline{\gamma} = \bigcup_{(i,j)\in I^2,\, i\ne j} \overline{\Omega_i} \cap \overline{\Omega_j}$ *is polygonal and has a non zero measure in* $\mathbb{R}^{d-1}$*.*
The meshes $(\mathcal{T}_i)_{i\in I}$ *are admissible finite volume meshes of the subdomains* $(\Omega_i)_{i\in I}$*, as*

defined in Definition 2.1. Let $\mathcal{T} = \bigcup_{i\in I} \mathcal{T}_i$ *be the finite volume mesh of* Ω, *which is, in general, atypical. If* $\mathcal{E}$ *is set of the edges of the control volumes of* $\mathcal{T}$, *we define* $\mathcal{E}_\gamma \subset \mathcal{E}$ *by* $\mathcal{E}_\gamma = \{\sigma \in \mathcal{E};\, \sigma \subset \gamma\}$.

Let us now give the discrete iterative domain decomposition procedure.

Definition 4.1 (The finite volume Schwarz algorithm)
Let $\alpha > 0$. *Let* $\left(u_K^{(0)}\right)_{K\in\mathcal{T}}$ *be a given vector of* $\mathbb{R}^{\mathrm{card}(\mathcal{T})}$ *and let* $\left(\left\{u_{K,\sigma}^{(0)}, u_{L,\sigma}^{(0)}\right\}\right)_{\sigma\in\mathcal{E}_\gamma}$ *be a given set of values associated with each edge* $\sigma \in \mathcal{E}_\gamma$, *one by side. At iteration* $n \geq 0$, *we assume the quantities* $\left(u_K^{(n)}\right)_{K\in\mathcal{T}}$ *and* $\left(\left\{u_{K,\sigma}^{(n)}, u_{L,\sigma}^{(n)}\right\}\right)_{\sigma\in\mathcal{E}_\gamma,\sigma=K|L}$ *to be known. For any control volume* K *neighbouring* $\mathcal{E}_\gamma$, *and* $\sigma \in \mathcal{E}_\gamma \cap \mathcal{E}_K$, *let*

$$F_{K,\sigma}^{(n)} = -\frac{\mathrm{m}(\sigma)}{\delta_{K,\sigma}}\left(u_{K,\sigma}^{(n)} - u_K^{(n)}\right), \tag{4.38}$$

and let

$$\Phi_{K,\sigma}^{(n)} = F_{K,\sigma}^{(n)} + \alpha\,\mathrm{m}(\sigma)\,u_{K,\sigma}^{(n)}, \tag{4.39}$$

and $F_{L,\sigma}^{(n)}$, $\Phi_{L,\sigma}^{(n)}$ *are defined by the same equality with* L *instead of* K. *Then,* $\left(u_K^{(n+1)}\right)_{K\in\mathcal{T}}$ *is sought as the solution to the following linear system of equations:*

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{(n+1)} + \sum_{\sigma\in\mathcal{E}_K} V_{K,\sigma}^{(n+1)} + b_K\,\mathrm{m}(K)u_K^{(n+1)} = \mathrm{m}(K)f_K,\ \forall K\in\mathcal{T}, \tag{4.40}$$

where the discrete diffusive flux $F_{K,\sigma}^{(n+1)}$ *is defined by:*

$$F_{K,\sigma}^{(n+1)} = -\frac{\mathrm{m}(\sigma)}{\delta_{K,\sigma}}\,(u_{K,\sigma}^{(n+1)} - u_K^{(n+1)}),\ \forall\sigma\in\mathcal{E}_K,\ \forall K\in\mathcal{T}, \tag{4.41}$$

with $u_{K,\sigma}^{(n+1)}$ *computed as in (2.6)-(2.7) if* $\sigma \in \mathcal{E}_{\mathrm{int}}\backslash\mathcal{E}_\gamma$, *and* $u_{K,\sigma}^{(n+1)}$ *computed by the following Robin condition if* $\sigma\in\mathcal{E}_\gamma$, $\sigma = K|L$:

$$-F_{K,\sigma}^{(n+1)} + \alpha\,\mathrm{m}(\sigma)\,u_{K,\sigma}^{(n+1)} = \Phi_{L,\sigma}^{(n)} = F_{L,\sigma}^{(n)} + \alpha\,\mathrm{m}(\sigma)\,u_{L,\sigma}^{(n)}. \tag{4.42}$$

Note that, for all $n \geq 0$, *we have*

$$u_{K,\sigma}^{(n+1)} = \frac{\delta_{K,\sigma}}{1+\alpha\,\delta_{K,\sigma}}\,\Phi_{L,\sigma}^{(n)} + \frac{1}{1+\alpha\,\delta_{K,\sigma}}\,u_K^{(n+1)}. \tag{4.43}$$

The convective term $V_{K,\sigma}^{(n+1)}$ is computed as in (2.8)-(2.9) if $\sigma \in \mathcal{E}\backslash\mathcal{E}_\gamma$, and

$$V_{K,\sigma}^{(n+1)} = v_{K,\sigma}^{+}\, u_K^{(n+1)} - v_{K,\sigma}^{-}\, u_L^{(n)} \;\; \textit{if } \sigma \in \mathcal{E}_\gamma, \;\; \textit{and } \sigma = K|L, \tag{4.44}$$

where $v_{K,\sigma}$ is defined by (2.9).
Finally, f_K and b_K are defined as in (2.4).

■

Remark 4.1 (Parallelization) *It is clear that the interface conditions (4.42) allow independent solves of the systems of linear equation by subdomain Ω_i, for $i \in I$, thus leading to an easy parallelization.*

Theorem 4.1 (Existence and uniqueness)
For given values of $\left(u_K^{(n)}\right)_{K\in\mathcal{T}} \in \mathbb{R}^{\mathrm{card}(\mathcal{T})}$ and $\left(\left\{\Phi_{K,\sigma}^{(n)}, \Phi_{L,\sigma}^{(n)}\right\}\right)_{\sigma\in\mathcal{E}_\gamma} \in \mathbb{R}^{\mathrm{card}(\mathcal{E}_\gamma)}$, Problem (4.40)-(4.44) has a unique solution in $\mathbb{R}^{\mathrm{card}(\mathcal{T})}$.

The proof follows that of theorem Proposition 5.1 in [Gallouët *et al.* 2000].

4.1 Convergence of the algorithm

Theorem 4.2 (Convergence) *Under Assumptions 2.1 and 4.1, let $\left(u^{(n)_K}\right)_{K\in\mathcal{T},n\in\mathbb{N}}$ be the sequence of vectors of $\mathbb{R}^{\mathrm{card}(\mathcal{T})}$ defined by the finite volume Schwarz algorithm of Definition 4.1. Let $(u_K)_{K\in\mathcal{T}}$ be the unique solution to Problem (2.3)-(2.9). Then for any $K \in \mathcal{T}$, the sequence $\left(u_K^{(n)}\right)_{n\in\mathbb{N}}$ converges to u_K in $\mathbb{R}$. Let $u_{\mathcal{T}}^{(n)} \in X(\mathcal{T})$, defined by $u_{\mathcal{T}}(x) = u_K$ and $u_{\mathcal{T}}^{(n)}(x) = u_K^{(n)}$ a.e. in K, for all $K \in \mathcal{T}$, then $\left(u_{\mathcal{T}}^{(n)}\right)_{n\in\mathbb{N}}$ converges towards $u_{\mathcal{T}}$ in $L^p(\Omega)$ for any $p \in [1, +\infty]$.*

Proof of Theorem 4.2

The proof of convergence is an adaptation to the discrete setting of the energy method used by P.L. Lions in [Lions 1990] for the continuous problem. Let $(u_K)_{K\in\mathcal{T}}$ be the solution of Problem (2.3)-(2.9) (that is the finite volume approximation of Problem (1.1)) for the mesh $\mathcal{T}$. One can remark that, since the problems are linear, the values $\left(u_K^{(n+1)} - u_K\right)_{K\in\mathcal{T}}$ satisfy all the equations (4.40)-(4.44) of the discrete Schwarz algorithm, with $f_K = 0$ and $g = 0$, for all $n \geq 0$. Indeed, for $(u_K)_{K\in\mathcal{T}}$, thanks to the conservativity of the discrete diffusive flux (see (2.6)), the values u_σ satisfy:

$$-F_{K,\sigma} + \alpha\,\mathrm{m}(\sigma)\,u_\sigma = F_{L,\sigma} + \alpha\,\mathrm{m}(\sigma)\,u_\sigma,$$

in particular for any $\sigma \in \mathcal{E}_\gamma$, $\sigma = K|L$. Hence the values $\left(u_K^{(n+1)} - u_K\right)_{K\in\mathcal{T}}$ verify the Robin conditions (4.42) on γ. We may therefore restrict our study to the case $f = 0$ and $g = 0$ and prove that the sequence $\left(u_K^{(n)}\right)_{n\in\mathbb{N}}$ defined by the discrete algorithm of Definition 4.1 converges towards 0, for all $K \in \mathcal{T}$. Let us multiply (4.40) by $u_K^{(n+1)}$ and sum over $K \in \mathcal{T}$. We deduce

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{(n+1)} u_K^{(n+1)} + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} V_{K,\sigma}^{(n+1)} u_K^{(n+1)} + \sum_{K\in\mathcal{T}} b_K\,\mathrm{m}(K)\,|u_K^{(n+1)}|^2 = 0. \tag{4.45}$$

We denote

$$\begin{aligned}
A^{(n+1)} &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{(n+1)} u_K^{(n+1)},\\
B^{(n+1)} &= \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} V_{K,\sigma}^{(n+1)} u_K^{(n+1)},\\
C^{(n+1)} &= \sum_{K\in\mathcal{T}} \mathrm{m}(K) b_K\,|u_K^{(n+1)}|^2.
\end{aligned}$$

In the sequel, for any $\sigma \in \mathcal{E}_{\mathrm{int}}$, we denote $\sigma = K_+|K_-$, with K_+ such that $v_{K_+,\sigma} \geq 0$. Note that $v_{K_-,\sigma} = -v_{K_+,\sigma}$. For all $\sigma \in \mathcal{E}_{\mathrm{ext}}$, we denote K the unique control volume of $\mathcal{T}$ of which σ is an edge.

• **Evaluation of** $A^{(n+1)}$ Reordering the summation over the set of the edges $\mathcal{E}$, using the definition of the set $\mathcal{E}_\gamma$, using (4.41), one deduces that

$$A^{(n+1)} = D^{(n+1)} + \sum_{\sigma\in\mathcal{E}_\gamma}\left(F_{K_+,\sigma}^{(n+1)} u_{K_+,\sigma}^{(n+1)} + F_{K_-,\sigma}^{(n+1)} u_{K_-,\sigma}^{(n+1)}\right),$$

with

$$\begin{aligned}
D^{(n+1)} = &\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\setminus\mathcal{E}_\gamma} \frac{\mathrm{m}(\sigma)}{\delta_\sigma}\left|u_{K_-}^{(n+1)} - u_{K_+}^{(n+1)}\right|^2 + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}_K} \frac{\mathrm{m}(\sigma)}{\delta_\sigma}|u_K^{(n+1)}|^2\\
&+ \sum_{\sigma\in\mathcal{E}_\gamma}\left(\frac{\mathrm{m}(\sigma)}{\delta_{K_+,\sigma}}\left|u_{K_+,\sigma}^{(n+1)} - u_{K_+}^{(n+1)}\right|^2 + \frac{\mathrm{m}(\sigma)}{\delta_{K_-,\sigma}}\left|u_{K_-,\sigma}^{(n+1)} - u_{K_-}^{(n+1)}\right|^2\right).
\end{aligned} \tag{4.46}$$

Note that $D^{(n)} \geq 0$, $\forall n \in \mathbb{N}$. Now following Lions' method in [Lions 1990], we remark that:

$$\sum_{\sigma\in\mathcal{E}_\gamma} F_{K_+,\sigma}^{(n+1)} u_{K_+,\sigma}^{(n+1)} = \frac{1}{4\alpha}\left(\sum_{\sigma\in\mathcal{E}_\gamma} \frac{1}{\mathrm{m}(\sigma)}\left(F_{K_+,\sigma}^{(n+1)} + \alpha\,\mathrm{m}(\sigma)\,u_{K_+,\sigma}^{(n+1)}\right)^2 - \sum_{\sigma\in\mathcal{E}_\gamma} \frac{1}{\mathrm{m}(\sigma)}\left(-F_{K_+,\sigma}^{(n+1)} + \alpha\,\mathrm{m}(\sigma)\,u_{K_+,\sigma}^{(n+1)}\right)^2\right),$$

and that the same result holds for K_- instead of K_+; using (4.42), that is to say

$$-F_{K_+,\sigma}^{(n+1)} + \alpha\,\mathrm{m}(\sigma)\,u_{K_+,\sigma}^{(n+1)} = F_{K_-,\sigma}^{(n)} + \alpha\,\mathrm{m}(\sigma)\,u_{K_-,\sigma}^{(n)},$$

and the same equality for K_- instead of K_+, we obtain:

$$A^{(n+1)} = D^{(n+1)} + E^{(n+1)} - E^{(n)}, \tag{4.47}$$

where, for all $n \geq 0$,

$$E^{(n)} = \frac{1}{4\alpha}\sum_{\sigma\in\mathcal{E}_\gamma} \frac{1}{\mathrm{m}(\sigma)}\left(F_{K_+,\sigma}^{(n)} + \alpha\,\mathrm{m}(\sigma)\,u_{K_+,\sigma}^{(n)}\right)^2 + \frac{1}{4\alpha}\sum_{\sigma\in\mathcal{E}_\gamma} \frac{1}{\mathrm{m}(\sigma)}\left(F_{K_-,\sigma}^{(n)} + \alpha\,\mathrm{m}(\sigma)\,u_{K_-,\sigma}^{(n)}\right)^2. \tag{4.48}$$

Note that $E^{(n)} \geq 0$, $\forall n \in \mathbb{N}$.

• **Evaluation of** $B^{(n+1)}$

Reordering the summation over the set of the edges of $\mathcal{E}$, using the upstream choice (4.44), and remarking that, as $g_\sigma = 0$, $V_{K,\sigma}\,u_K^{(n+1)} \geq \frac{1}{2}\,v_{K,\sigma}\left|u_K^{(n+1)}\right|^2$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$, one deduces that

$$B^{(n+1)} \geq B_1^{(n+1)} + B_2^{(n+1)}, \tag{4.49}$$

with

$$B_1^{(n+1)} = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\setminus\mathcal{E}_\gamma} v_{K_+,\sigma}\left(u_{K_+}^{(n+1)} - u_{K_-}^{(n+1)}\right) u_{K_+}^{(n+1)} + \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}} v_{K,\sigma}\left|u_K^{(n+1)}\right|^2,$$

and

$$B_2^{(n+1)} = \sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n+1)}\right|^2 - \sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\,u_{K_+}^{(n)}\,u_{K_-}^{(n+1)}.$$

Reordering the summation over $K \in \mathcal{T}$ the first term of the sum, using the estimate $(a-b)\,a \geq \frac{1}{2}\,a^2 - \frac{1}{2}\,b^2$ and remarking that $v_{K_-,\sigma} = -v_{K_+,\sigma}$ yield

$$\sum_{\sigma\in\mathcal{E}_{\rm int}} v_{K_+,\sigma}\left(u_{K_+}^{(n+1)} - u_{K_-}^{(n+1)}\right) u_{K_+}^{(n+1)} \geq \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}} v_{K,\sigma}|u_K^{(n+1)}|^2.$$

It gives us

$$B_1^{(n+1)} \geq \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}\left|u_K^{(n+1)}\right|^2 - \frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma}\left(v_{K_+,\sigma}\left|u_{K_+}^{(n+1)}\right|^2 - v_{K_+,\sigma}\left|u_{K_-}^{(n+1)}\right|^2\right).$$

Using now the definition of $v_{K,\sigma}$ and the regularity of $\boldsymbol{v}$, we obtain

$$\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma} = \int_K \mathrm{div}(\boldsymbol{v}(x))dx.$$

Furthermore, remarking that

$$C^{(n+1)} = \sum_{K\in\mathcal{T}}\int_K b(x)\,dx,$$

and using the assumption, $\frac{1}{2}\mathrm{div}\boldsymbol{v}(x) + b(x) \geq 0$ a.e on Ω, yield

$$\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}|u_K^{(n+1)}|^2 + C^{(n+1)} \geq 0.$$

We deduce that

$$B_1^{(n+1)} + C^{(n+1)} \geq -\frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n+1)}\right|^2 + \frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_-}^{(n+1)}\right|^2. \tag{4.50}$$

Using the estimate $a\,b = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$ we obtain

$$B_2^{(n+1)} = \sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n+1)}\right|^2 - \frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n)}\right|^2$$

$$-\frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K,\sigma}\left|u_{K_-}^{(n+1)}\right|^2 + \frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n)} - u_{K_-}^{(n+1)}\right|^2.$$

As $v_{K_+,\sigma} \geq 0$, it gives

$$B_2^{(n+1)} \geq \sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n+1)}\right|^2 - \frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_+}^{(n)}\right|^2 - \frac{1}{2}\sum_{\sigma\in\mathcal{E}_\gamma} v_{K_+,\sigma}\left|u_{K_-}^{(n+1)}\right|^2.$$

Using now the previous estimate, using (4.49) and (4.50) yield

$$B^{(n+1)} + C^{(n+1)} \geq F^{(n+1)} - F^{(n)}, \tag{4.51}$$

with

$$F^{(n)} = \frac{1}{2} \sum_{\sigma \in \mathcal{E}_\gamma} v_{K_+,\sigma} \left| u_{K_+}^{(n)} \right|^2,$$

for all $n \geq 0$. Note that $F^{(n)} \geq 0$, $\forall n \geq 0$.

• **End of the proof**

Using (4.45), (4.47) and (4.51) yields

$$D^{(n+1)} + E^{(n+1)} - E^{(n)} + F^{(n+1)} - F^{(n)} \leq 0.$$

Let us sum from $n = 0$ to $n = N$, we have

$$\sum_{n=0}^{N+1} D^{(n+1)} + E^{(N+1)} + F^{(N+1)} \leq E^{(0)} + F^{(0)}.$$

As $E^{(N+1)} + F^{(N+1)} \geq 0$ we obtain

$$\sum_{n=0}^{N+1} D^{(n+1)} \leq E^{(0)} + F^{(0)}. \tag{4.52}$$

We deduce from (4.52) that the non negative series $\sum_{n=0}^{N+1} D^{(n+1)}$ converges in $\mathbb{R}$, and therefore the sequence $\left(D^{(n)}\right)_{n\in\mathbb{N}}$ tends to 0 in $\mathbb{R}$ as n tends to infinity.

Using the definition of $D^{(n)}$, (4.46) and the Robin conditions (4.43), we deduce the following convergence results:

- for all $K \in \mathcal{T}$ such that $\mathrm{m}(\partial K \cap \partial\Omega) > 0$, the sequence $\left(u_K^{(n)}\right)_{n\in\mathbb{N}}$ converges towards 0 in $\mathbb{R}$,
- for all $(K_1, K_2) \in \mathcal{T}_i$, for all $i \in I$, the sequence $\left(u_{K_1}^{(n)} - u_{K_2}^{(n)}\right)_{n\in\mathbb{N}}$ converges towards 0 in $\mathbb{R}$,
- if $K \in \mathcal{T}_i$ and $L \in \mathcal{T}_j$ are such that $\sigma = K|L$ is in $\mathcal{E}_\gamma$, then

* the sequence $\left(u_K^{(n)} - u_{K,\sigma}^{(n)}\right)_{n\in\mathbb{N}}$ converges towards 0 in $\mathbb{R}$,
* the sequence $\left(u_L^{(n)} - u_{L,\sigma}^{(n)}\right)_{n\in\mathbb{N}}$ converges towards 0 in $\mathbb{R}$,
* the sequence $\left(u_{L,\sigma}^{(n+1)} - u_{K,\sigma}^{(n)}\right)_{n\in\mathbb{N}}$ converges towards 0 in $\mathbb{R}$. The same result holds with L instead of K.

Let $K \in \mathcal{T}_i$, $K \subset \Omega_i$. Thanks to the path-connectivity of Ω, there exists a finite sequence of interfaces $(\sigma_k)_{1\leq k\leq N}$, with $N \leq \text{card}(I)$ such that:

* $\sigma_N \in \mathcal{E}_{\text{ext}}$ and σ_N is an edge of K_N. We have $\text{m}(\partial K_N \cap \partial\Omega) > 0$
* $\sigma_k \in \mathcal{E}_\gamma$ for all $1 \leq k \leq N-1$, with $\sigma_k = K_k|L_k$, where $K_1 \subset \Omega_i$ and K_{k+1} and L_k are in the same subdomain Ω_{i_k}, for $1 \leq k \leq N-1$.

Remarking now that

$$\left|u_K^{(n)}\right| \leq \left|u_K^{(n)} - u_{K_1}^{(n)}\right| + \left|u_{K_1}^{(n)} - u_{K_1,\sigma_1}^{(n)}\right|$$

$$+ \left|u_{K_1,\sigma_1}^{(n)} - u_{L_1,\sigma_1}^{(n+1)}\right| + \left|u_{L_1,\sigma_1}^{(n+1)} - u_{L_1}^{(n+1)}\right| + \left|u_{L1}^{(n+1)} - u_{K_2}^{(n+1)}\right| +$$

$$\cdots + \left|u_{K_{N-1},\sigma_{N-1}}^{(n+N-1)} - u_{L_{N-1},\sigma_{N-1}}^{(n+N)}\right| + \left|u_{L_{N-1},\sigma_{N-1}}^{(n+N)} - u_{L_{N-1}}^{(n+N)}\right|$$

$$+ \left|u_{L_{N-1}}^{(n+N)} - u_{K_N}^{(n+N)}\right| + \left|u_{K_N}^{(n+N)}\right|,$$

and using the previous convergence results, we can prove that the sequence $\left(u_K^{(n)}\right)_{n\in\mathbb{N}}$ converges towards 0 in $\mathbb{R}$, which concludes the proof. ■

Remark 4.2 (d_σ instead of δ_σ)
If we use d_σ instead of δ_σ in the definition of the discrete diffusive fluxes(4.41), we obtain the same convergence result for the finite volume Schwarz algorithm.

5 Numerical tests

We illustrate in this section the convergence of the schemes presented above. We prove by numerical experiments that the convergence obtained for the FVD scheme is optimal in 2D. We then show how fast the convergence of the Schwarz algorithm is and emphasize the advantage of a true Robin condition at the interface by studying the dependence on α of the iterative algorithm. Let Ω be decomposed into rectangular subdomains $\Omega = \cup_{i=1}^{N} \Omega_i$. We shall present simulations for $N = 2, 3, 9$. Each subdomain is assigned with a uniform or a non uniform rectangular mesh $\mathcal{T}_i^h$ independently from one another as shown in Figure 4. We can see that the mesh $\mathcal{T}^h = \cup_{i=1}^{N} \mathcal{T}_i^h$ may feature atypical edges located at the

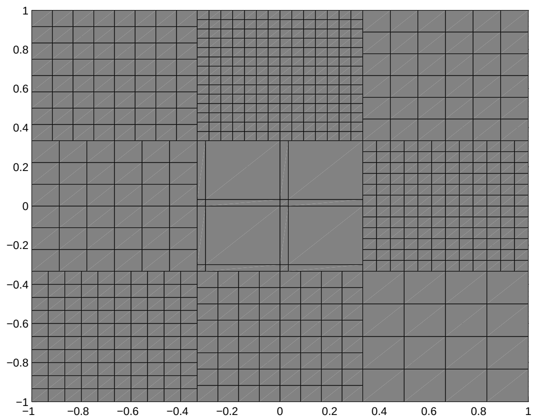


Figure 4: Example of mesh

interfaces $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. The family of meshes $\mathcal{T}^h$ may satisfy the regularity conditions (3.29). Let us give an example of such a mesh: for a uniform mesh, the family of mesh $(\mathcal{T}_i^h)$ is given by a step size in the two dimensions :

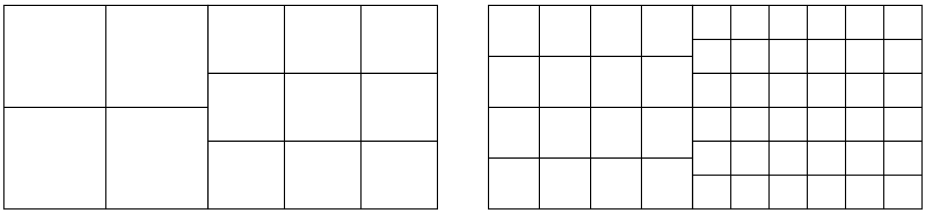
$$\mathcal{T}_i^h = (\Delta x_i^h, \Delta y_i^h) = (h\Delta x_i^0, h\Delta y_i^0),$$

where $\mathcal{T}_i^0 = (\Delta x_i^0, \Delta y_i^0)$ denotes a given initial mesh. It is then easy to prove that

$$\exists \zeta > 0,\, \zeta \max(\Delta x_i^h, \Delta y_i^h) \leq d_{K,\sigma}.$$

We first consider smooth solutions of the following problem

$$-\varepsilon \Delta u + \operatorname{div}(\boldsymbol{v}\, u) + b\, u = f. \tag{5.53}$$



$\mathcal{T}^0 = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3})$ $\mathcal{T}^{\frac{1}{2}} = (\frac{1}{4}, \frac{1}{4}), (\frac{1}{6}, \frac{1}{6})$

Figure 5: Example of a family of meshes

- Case n^{o}1 : Pure diffusion operator, Homogeneous Dirichlet Boundary Conditions

$$u(x,y) = \sin(\pi x)\sin(\pi y) \text{ for } v(x,y) = 0,\ b = 1,\ \varepsilon = 1$$

- Case n^{o}2: Pure diffusion operator, Non Homogeneous Dirichlet Boundary Conditions

$$u(x,y) = \cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) \text{ for } v(x,y) = 0,\ b = 1,\ \varepsilon = 1$$

- Case n^{o}3 : Convection Diffusion operator, Stiff Case, Non Homogeneous Dirichlet Boundary Conditions

$$u(x,y) = \sin(\frac{1}{2}(x+1)(y+1)) + \frac{1}{4}(x+1)^3(y+1)^2$$

  for

$$v(x,y) = ((y+1)/2, -(x+1)),\ b = 1,;\varepsilon = 0.01 \text{ or } \varepsilon = 1$$

  (This case is also studied in [Achdou *et al.* 2002].)

The source term f is chosen such that u is solution to the problem (5.53).
We also propose a weak solution of problem (5.53) that is $u \notin H^2$ on a non convex domain as shown in Figures 7 or 9 :

- Case n^{o}4 : $u(x,y) = \tilde{u}(r,\theta) = r^{\frac{2}{3}}\sin(\frac{2}{3}(\theta + \frac{\pi}{2}))$ for $v(x,y) = 0,\ b = 0,\ f = 0.$

In Figure 6 for standard meshes, we get the classical order 1 convergence in H^1 norm for smooth solution, the super convergence order 2 in H^1 norm for pure diffusion operator and homogeneous Dirichlet conditions is also obtained. This is in accordance with the theoritical results obtained for cartesian meshes, see [Gallouët *et al.* 2000]. Figure 7 shows that if $u \in H^s$ ($s = \frac{5}{3}$ in Case 4) the usual $s-1$ convergence order ($\frac{2}{3}$ in Case 4) obtained in the finite element method framework [Ciarlet 1990] also holds for the finite volume scheme, as shown in [Droniou & Gallouët 2002]. Figures 8 and 9 show that the order

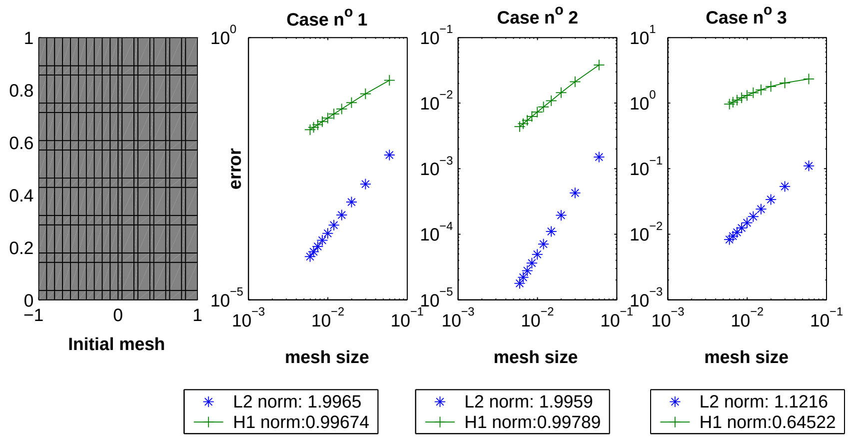


Figure 6: Convergence order of the FVD scheme for a regular solution on a regular mesh

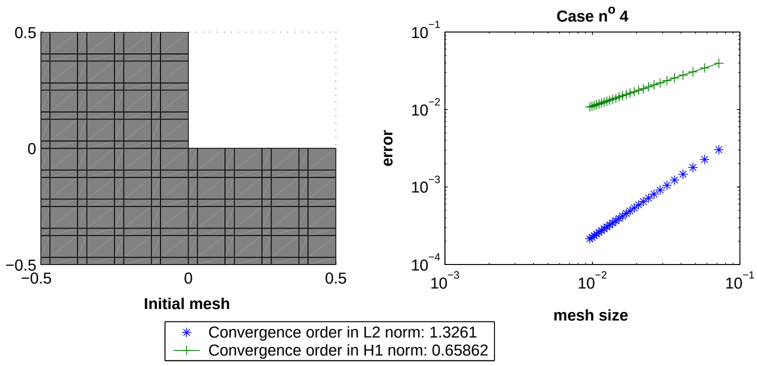


Figure 7: Convergence order of the FVD scheme for a $H^{\frac{5}{3}}$ solution on a regular mesh

$\frac{1}{2}$ in H^1 norm is optimal as soon as the mesh presents atypical edges. We next present

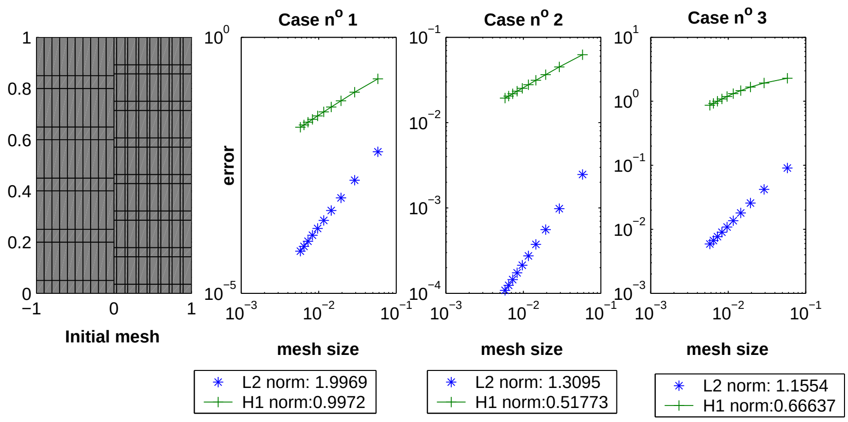


Figure 8: Convergence order of the FVD scheme for a regular solution on an atypical mesh

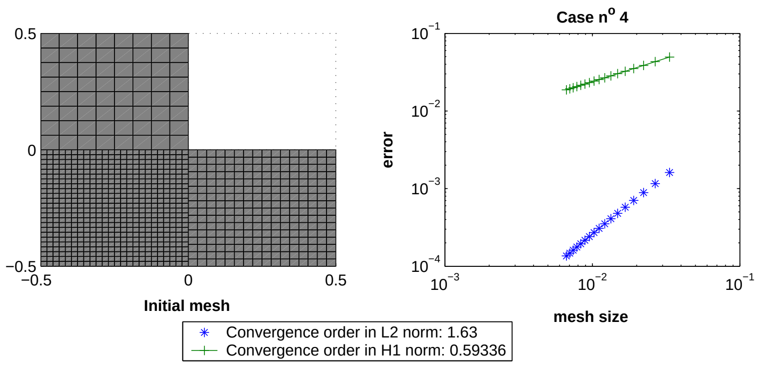


Figure 9: Convergence order of the FVD scheme for a $H^{\frac{5}{3}}$ solution on an atypical mesh

the results of the Schwarz algorithm in the case of 9 subdomains. We stop the Schwarz algorithm if

$$\|u_D - u_S\|_\infty \leq \|u_D - u\|_\infty$$

where u_D, u_S, u denote respectively the solutions of the FVD scheme, the Schwarz algorithm and the exact solution of problem (5.53). Figure 10 and 11 shows that we need few iterations to be as good as the direct scheme. We also remark that when α goes to zero; the number of iterations increases and $\|u_D - u_S\|_\infty$ has a minimun located between 0.5 and 1.

6 Conclusion

In this paper we showed that the finite volume method may be used with a non-conforming mesh and still yield an order of convergence of at least 1/2 under reasonable assumptions on the non matching interface and for a H^2 regularity of the exact solution. There are several ways to improve the accuracy of the method by modifying the computation of the fluxes at the non matching interfaces so as to obtain a consistent expression. One may use for instance the "nine points" scheme which is used in oil engineering codes for non rectangular grids or heterogeneous media (see [Faille 1992] or [Eymard *et al.* 2000]). One may also use the mortar technique recently adapted by F. Nataf *et al.* [Achdou *et al.* 2002], [Saas *et al.* 2002]. However in both cases, it seems to be difficult to prove the maximum principle, since the approximation of the diffusion flux no longer writes under the form (2.16); one is therefore faced with the usual "accuracy vs. stability" dilemna. The Lions-Schwarz algorithm is well adapted to the discrete finite volume setting, and the proof of convergence was transposed to the discrete setting. One of the main advantages of the Robin interface conditions is that they yield a non overlapping domain decomposition method. However, in the actual implementation, the coefficient α needs to be tuned. In [Gander *et al.* 2002] a study of an optimal alpha was performed in the case of a pure diffusion operator, and for a homogeneous media. Here we used the decomposition method as a pure iterative method. Of course, it can also be used in as a preconditioner in a conjugate gradient, or GMRES [Saad & Schultz 1986] in the case of a convection diffusion operator, in order to improve the speed of convergence.

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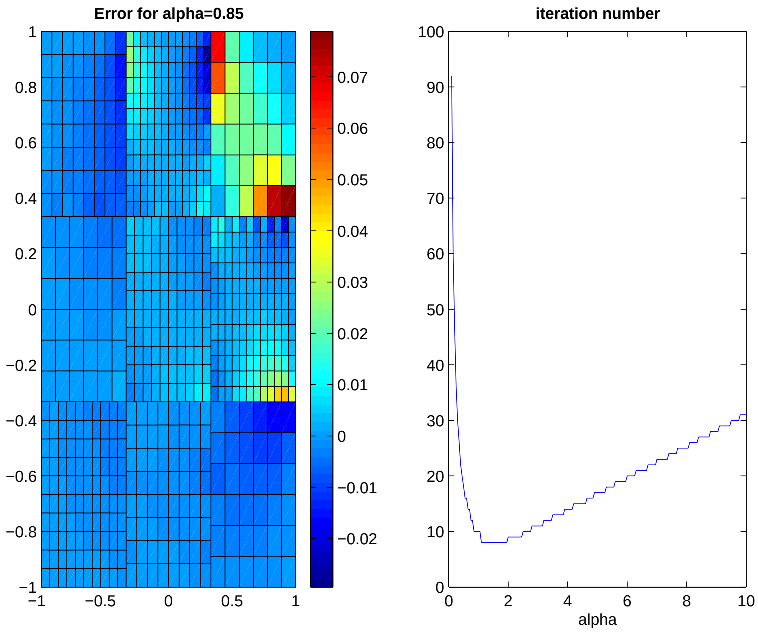

Figure 10: Error estimate between the Schwarz algorithm and the FVD scheme for Case n^{o}3, $\varepsilon = 1$.

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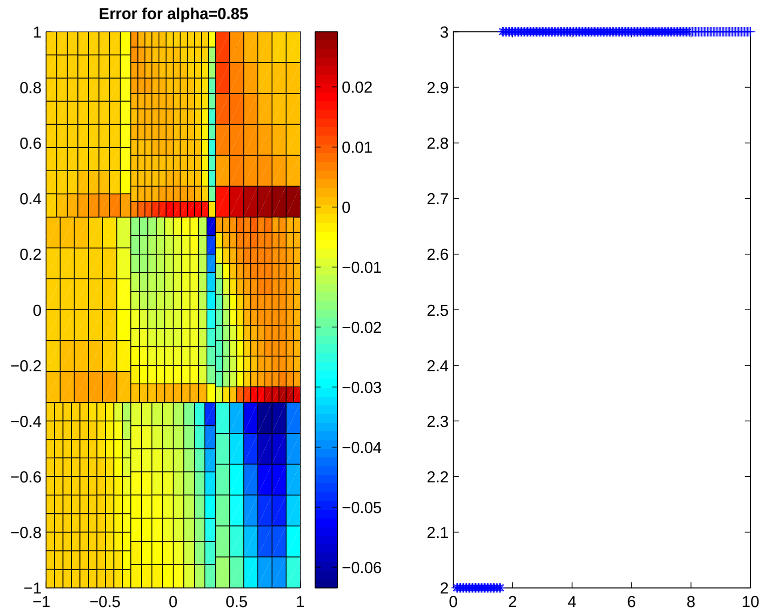

Figure 11: Error estimate between the Schwarz algorithm and the FVD scheme for Case n^{o}3,$\varepsilon = 0.01$.

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